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Boson Expansion Methods in (1+1)-dimensional Light-Front QCD

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Abstract

We derive a bosonic Hamiltonian from two dimensional QCD on the light-front. To obtain the bosonic theory we find that it is useful to apply the boson expansion method which is the standard technique in quantum many-body physics. We introduce bilocal boson operators to represent the gauge-invariant quark bilinears and then local boson operators as the collective states of the bilocal bosons. If we adopt the Holstein-Primakoff type among various representations, we obtain a theory of infinitely many interacting bosons, whose masses are the eigenvalues of the 't Hooft equation. In the large N limit, since the interaction disappears and the bosons are identified with mesons, we obtain a free Hamiltonian with infinite kinds of mesons.

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1 Introduction

The low energy dynamics of strong interactions is usually investigated in the framework of effective theories introduced phenomenologically [1]. Now that the strong interaction is believed to be described by quantum chromodynamics (QCD), these effective theories, if they are correct, should be derived from QCD in some way or another. For example, although the chiral perturbation theory or the Skyrme model describes the low energy physics involving the Nambu-Goldstone (NG) bosons, we cannot determine definitely the forms of interactions or the magnitude of the parameters. Then a natural question arises of how to obtain directly such a hadronic effective theory from QCD, which may be answered after solving the problem of confinement and dynamical chiral symmetry breaking. For the realistic case, QCD_{3+1}, this problem is still far beyond our knowledge. Two-dimensional QCD, however, may serve as a tractable model at least to know some consequences after confinement.

Two dimensional QCD occupies a very unique position as a toy model for the 4 dimensional QCD. Originally it was introduced by 't Hooft [2] to demonstrate that we can sum up the planar diagrams and can get nonperturbative results in the limit of a large number of colors (the large N limit). The Bethe-Salpeter equation for the quark-antiquark system becomes simple and leads to what is now called the 't Hooft equation which determines the mesonic spectra in the large N limit. Second, two dimensional gauge theories have a linear potential and, thus, are confining in the lowest perturbation theory. Also, nonperturbatively, 't Hooft showed that the mesonic spectra are discrete, which suggests the confinement of quarks in the large N limit. Third, Kikkawa [3] presented the idea that the Hamiltonian can be easily rewritten only in terms of a gauge-invariant bilocal operator due to the absence of propagating degrees of freedom in two dimensional gauge theories. This operator is a path-ordered product with its ends attached by a quark and an anti-quark, and thus physically close to a mesonic state. Several works using gauge-invariant bilocal operators have been performed in two dimensional QCD [4, 5, 6, 7, 8, 9]. From these facts, two dimensional QCD can be understood as an appropriate model to investigate how hadronic theory can be constructed after confinement and to discuss the higher order effects of the $1/N$ expansion, which is necessary for the realistic case $N = 3$.

Since both of the works above greatly benefited from the light-front (LF) frame, we also work on the light-front to go beyond them. Recently the light-front Hamiltonian field theory has acquired renewed interest with the hope of solving nonperturbatively the relativistic bound-state problems [10, 11, 12, 13, 14]. Its main strategy is the "trivial" vacuum and the approximation to a restricted Fock space with a few number of particles. Bound states are

analyzed within a subspace with a few numbers of particles. In this sense, this approach is similar to the constituent quark picture, where the dynamical variables are a few constituent quarks. On the LF, a careful treatment of the zero mode is important for an understanding of spontaneous symmetry breaking (for example, see [15, 16]) and vacuum structures such as the θ vacuum, but we do not consider the zero modes in this paper and we will work on the infinite light-front space. This is reasonable because in two dimensions there is no spontaneous symmetry breaking for *finite* N theories [17] and also there is no topological configuration in $SU(N)$ or $U(N)$ QCD_{1+1} with fundamental fermions. According to the calculations of the various two-dimensional models [18, 19, 20, 21, 22, 23, 24], we know that the Tamm-Dancoff approximation on the light-front (Light-Front Tamm-Dancoff approach, LFTD) [25, 26, 27] is very good for lower excitations. Especially the lowest meson state is found to be a two-body state (a quark-antiquark state) almost perfectly. Although the success of LFTD will be partially due to the triviality of the vacuum, why LFTD is so good is not thoroughly understood yet and it is not certain that we can also describe the NG bosons in this approximation. Indeed, in the constituent quark model, pions cannot be described by few constituents. Since there are no NG bosons in two dimensions, the success of LFTD in two dimensions does not tell anything about the NG bosons. Furthermore, a more fundamental question of whether we can describe the NG bosons on the LF is not so clearly answered to date [28]. Therefore approaches independent of the Tamm-Dancoff approximation will be needed to understand why LFTD is good and to go to higher dimensions where the NG mode may exist.

Motivated by these we introduce a method to obtain a bosonic Hamiltonian from fermionic systems. This is a well-known technique in quantum many-body physics and is called the boson expansion method or the boson mapping [29]. Using this method we can introduce bosonic variables to represent bifermion operators. On the other hand, the work of Kikkawa can be understood as the operator theory of the quark-antiquark bound states. The variable in the formalism is the bilocal operators. We extract bosonic variables from the bilocal operators and construct a bosonic theory. Since the main purpose of this paper is to indicate that the use of the boson expansion method is very natural to obtain a bosonic theory on the LF, its conceptual aspects are emphasized throughout the paper.

This paper is organized as follows. In the next section, the basics of QCD_{1+1} on the LF is presented and the gauge-invariant approach by Kikkawa is given. It is shown that an equation of motion for the two-body state becomes the 't Hooft equation in the large N limit. In Sec. III, we consider the general structures of the two-body states and why we need boson expansion methods is explained. In Sec. IV, the boson expansion methods are shortly reviewed and applied

to 2D QCD. Here we introduce bilocal boson operators to represent fermion bilinear operators. Local boson operators are given in Sec. V as collective bosons of the bilocal operators. Finally Sec. VI is devoted to a summary and discussion. The Appendix shows some relations for operators between the gauge-invariant approach and the usual one.

2 Light-front QCD_{1+1} and its bilocal formulation

In this section, we fix the notation for two dimensional QCD on the light front and introduce its bilocal formulation of Kikkawa [3]. This formalism gives a Hamiltonian only in terms of gauge-invariant bilocal operators, which are essentially quark bilinears. In the large N limit, the Heisenberg equation of motion for the bilinear operator becomes the 't Hooft equation [2].

2.1 QCD_{1+1} in light-front coordinates

The model we consider is a two-dimensional gauge theory with a massive quark in the fundamental representation of $SU(N)$ or $U(N)$. For simplicity we treat a one-flavor quark, but the generalization to many flavors is straightforward. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ is the field strength and $D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igA^a_\mu T^a$ is a covariant derivative. The $N \times N$ matrices T^a, generators of the group, are normalized as $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ and satisfy the Lie algebra $[T^a, T^b] = if^{abc}T^c$. The fundamental quark Ψ is a two-component spinor and we define

$$\Psi_i = 2^{-\frac{1}{4}}\begin{pmatrix} \psi_i \\ \chi_i \end{pmatrix}, \tag{2.2}$$

where $i = 1, \ldots, N$ is a color index. We use the following γ-matrices; $\gamma_0 = \sigma_1,\ \gamma_1 = i\sigma_2$.

We work in the light-front coordinates, $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, and thus regard x^+ as 'time' and so $\partial_+ = \frac{\partial}{\partial x^+}(= \partial^-)$ as 'time' derivative. To avoid confusion, we shall use only ∂_+ and the spatial derivative ∂_- instead of ∂^- or ∂^+. If we choose the light-front 'temporal' gauge, $A^+ = 0$, the Lagrangian becomes

$$\mathcal{L} = \mathrm{tr}(\partial_- A_+)^2 + i\psi^\dagger D_+\psi + i\chi^\dagger\partial_-\chi - \frac{m}{\sqrt{2}}(\psi^\dagger\chi + \chi^\dagger\psi). \tag{2.3}$$

It is evident that A_+ and χ are not dynamical, because there is no kinetic term for them in eq.(2.3). The constraints,

$$\begin{aligned} \partial_-^2 A_+^a - g\psi^\dagger T^a\psi &= 0, && (2.4) \\ i\partial_-\chi - \frac{m}{\sqrt{2}}\psi &= 0, && (2.5) \end{aligned}$$

are solved easily and the solutions are

$$A_+^a(x) = \frac{g}{2}\int_{-\infty}^{\infty} dy\ |x-y|J^a(y) + C^a x + D^a, \tag{2.6}$$

$$\chi(x) = \frac{m}{2\sqrt{2}i}\int_{-\infty}^{\infty} dy\ \epsilon(x-y)\psi(y), \tag{2.7}$$

where $J^a(x) = \psi^\dagger(x)T^a\psi(x)$ and $\epsilon(x)$ is a sign function $\epsilon(x) = x/|x|$.[2] As was pointed out by Bars and Green [30], we can set $C^a = D^a = 0$ in the physical space. And also, since we do not consider the fermionic zero mode, we did not write the boundary terms in eq. (2.7). Inserting these solutions into the Hamiltonian and using the same trick for the product of the sign functions as that of Kikkawa [3], we have

$$\begin{aligned} H &= -\frac{g^2}{4}\int dxdx'\ |x-x'|J^a(x)J^a(x') \\ &\quad +\frac{m^2}{4i}\int dxdx'\ \epsilon(x-x')\psi^\dagger(x)\psi(x'). \end{aligned} \tag{2.8}$$

The remnant of the gauge field can be found as the linear potential in the first term. Finally, a quantization condition is imposed on the dynamical variable ψ;

$$\begin{aligned} \{\psi_i(x), \psi_j^\dagger(y)\}_{x^+=y^+} &= \delta_{ij}\delta(x-y), \\ \{\psi_i(x), \psi_j(y)\}_{x^+=y^+} &= \{\psi_i^\dagger(x), \psi_j^\dagger(y)\}_{x^+=y^+} = 0, \end{aligned} \tag{2.9}$$

The Fock vacuum is

$$a^i_{k_-}|0> = d^i_{k_-}|0> = 0 \qquad (\forall i = 1, \ldots, N\ \ ,\ \ k_- > 0), \tag{2.10}$$

where annihilation operators are defined as

$$\psi_i(x) = \int_0^\infty \frac{dk_-}{\sqrt{2\pi}}\left\{a^i_{k_-}e^{-ikx} + d^{i\dagger}_{k_-}e^{ikx}\right\}. \tag{2.11}$$

On the LF, as long as we do not consider the zero mode, the Fock vacuum $|0>$ is also the vacuum of the interacting Hamiltonian (2.8).

2.2 Bilocal formulation–Gauge invariant variables

Following Kikkawa [3], let us introduce a color-singlet bilocal field and its Fourier transform;

$$M(x,y) = \frac{1}{2}\sum_{i=1}^{N}[\psi_i^\dagger(x), \psi_i(y)], \tag{2.12}$$

[2] From now on we write the light-front spatial variables without their suffices, i.e. x and p instead of x^- or p_-. Moreover, since we work in the Hamiltonian formalism, we omit the light-front 'time' coordinate.

$$M(p_1,p_2)=\int_{-\infty}^{\infty}\frac{dx_1}{\sqrt{2\pi}}\frac{dx_2}{\sqrt{2\pi}}\quad M(x_1,x_2)e^{ip_1x_1+ip_2x_2}. \tag{2.13}$$

Although this is defined as the commutator of fermion fields, it is not essential because we treat only normal ordered operators in the following discussions. If we choose the $A^-=0$ gauge as in Ref. [3], M is given by

$$\mathcal{M}(x,y)=\frac{1}{2}\left[\psi^\dagger(x)\mathrm{P}e^{ig\int_y^x A_-(z)dz^-}\quad,\quad\psi(y)\right], \tag{2.14}$$

where P denotes the path-ordered product. Since we are working at a fixed time in two dimensions, there is no path dynamics for the Wilson line. This operator is gauge equivalent to Eq.(2.12). Indeed if we perform the gauge transformation, $U(x)=\mathrm{P}\exp\{ig\int_x^{-\infty}A_-dz^-\}$, Eq.(2.14) is transformed into Eq.(2.12).

Quantization of $M(p_1,p_2)=M^\dagger(p_2,p_1)$ is given by the commutation relation;

$$\begin{aligned}&[:M(p_1,p_2):,:M(q_1,q_2):]\\ &=\ :M(p_1,q_2):\delta(p_2+q_1)-:M(q_1,p_2):\delta(p_1+q_2) \\ &+\ N\delta(p_1+q_2)\delta(q_1+p_2)\{\theta(p_1)\theta(p_2)\theta(-q_1)\theta(-q_2)-\theta(-p_1)\theta(-p_2)\theta(q_1)\theta(q_2)\}.\end{aligned} \tag{2.15}$$

This is derived from the anti-commutation relations (2.9). The normal order of M is defined through the quark operators (2.11) in terms of the Fock vacuum. The last c-number term that emerged from normal-ordering is very important for later discussions. Decomposing M into four parts by the sign of the momentum as

$$M_{\tau_1\tau_2}(p_1,p_2)=\theta(\tau_1p_1)\theta(\tau_2p_2)M(p_1,p_2), \tag{2.16}$$

we can see that only $:M_{--}:$ does not annihilate the vacuum;

$$:M_{++}(p_1,p_2):|0>=:M_{+-}(p_1,p_2):|0>=:M_{-+}(p_1,p_2):|0>=0. \tag{2.17}$$

The state $:M_{--}:|0>$ corresponds to the quark-antiquark two-body state.[3] Although the full Fock space is constructed by further introducing gauge-invariant baryon operators [3], we treat, in this paper, only the bosonic sector (i.e. a subspace with a zero fermion number);

$$\mathcal{F}_B=\left\{\prod_i(:M_{--}(p_i,q_i):)\,|0>\right\}. \tag{2.18}$$

The Hamiltonian (2.8) is rewritten by using the normal ordered bilocal fields as

$$H\ =\ -\frac{g^2}{8\pi}\int_{-\infty}^{\infty}dp_1dp_2dq_1dq_2\ \delta(p_1+p_2+q_1+q_2)\times$$

[3] We call this state as a "two-body state" instead of a "mesonic state" as in Ref. [3], because this is not an eigenstate of the Hamiltonian as we shall see.

$$
\left\{ \frac{\mathcal{P}}{(p_1+q_2)^2} + \frac{1}{N}\frac{\mathcal{P}}{(p_1+p_2)^2} \right\} : M(p_1,p_2) :: M(q_1,q_2) :
$$
$$
-\frac{1}{2}\left(m^2 - \frac{g^2 N}{2\pi}\right)\int_{-\infty}^{\infty} dq\ \frac{\mathcal{P}}{q} : M(q,-q) :, \tag{2.19}
$$

where $\mathcal{P}$ denotes the principal value prescription of the integral to avoid infrared divergences [2]. The term with $1/N$ in the first term is absent for the $U(N)$ theories. This Hamiltonian is not completely normal ordered, e.g. $: M_{+-} :: M_{--} :$ is not in the normal order. This is because we want to keep in touch with the unit $: M :$ which is close to the mesonic object. Although the Hamiltonian (2.19) is quadratic in terms of $: M :$, the Heisenberg equation of motion for the two-body state $: M_{--} : |0>$ has interaction terms due to the nontrivial algebra (2.15). For $U(N)$ theory (hereafter we treat only $U(N)$ case for simplicity) we obtain

$$
\begin{aligned}
i\partial_+ : M_{--}(q_1,q_2) : |0> &= [: M_{--}(q_1,q_2) :, \quad : H :]|0> \\
&= \frac{g^2 N}{4\pi}\int dk \frac{\mathcal{P}}{k^2} : M_{--}(q_1-k, q_2+k) : |0> \\
&+ \frac{1}{2}\left(m^2 - \frac{g^2 N}{2\pi}\right)\left(\frac{1}{q_1}+\frac{1}{q_2}\right) : M_{--}(q_1,q_2) : |0> \\
&- \frac{g^2}{4\pi}\int dk dp\ \frac{\mathcal{P}}{k^2}(: M_{--}(q_1, k-p) :: M_{--}(p, q_2-k) : \\
&\qquad - : M_{--}(p,q_2) :: M_{--}(q_1-k, k-p) :)|0> .
\end{aligned} \tag{2.20}
$$

Since it is reasonable to treat $: M_{\pm\pm} :$ and $: M_{\pm\mp} :$ to be $O(N^{1/2})$ and $O(N^0)$ respectively, we find that the first two terms give the leading contribution ($O(N^0)$) in the large N limit ($N \to \infty$ with $g^2 N$ fixed), and that the last term is in the next order. The last term represents the dissociation of a single two-body state into two two-body states. The leading contribution becomes a homogeneous equation and leads to the 't Hooft equation [2] which determines the mesonic spectra in the large N limit,

$$
\mu^2 \phi(z) = \left(m^2 - \frac{g^2 N}{2\pi}\right)\left(\frac{1}{z} + \frac{1}{1-z}\right)\phi(z) - \frac{N g^2}{2\pi}\int_0^1 dy \frac{\mathcal{P}}{(y-z)^2}\phi(y), \tag{2.21}
$$

where $\mu^2 = 2r_- r_+$ is the invariant mass, $r_- = q_1 + q_2$ is the total momentum, $z = q_1/r_-$ is the momentum fraction of a quark, and $\phi(z) =< 0|(: M_{--}(q_1,q_2) :)^\dagger|\phi>$ is the wave function of the relative motion. Therefore the two-body state can be identified with a meson in the large N limit. The meson does not dissociate into two mesons in leading order.

3 General consideration of two-body states

In the previous section, we learned from Kikkawa's formalism that even though we start from the Hamiltonian (2.19) quadratic in terms of the bilocal operators, the resulting Heisenberg

equation of motion (2.20) shows that there exists an interaction. In the large N limit, however, the interaction vanishes and thus the two-body state becomes free and is identified with the meson. The origin of the difference between finite and infinite N theories is that while the algebra (2.15) is nontrivial for finite N, it goes to a bosonic commutator in the large N limit ($: M_{++} :$ is the hermitian conjugate to $: M_{--} :$, see the Appendix);

$$\lim_{N\to\infty}[: M_{++}(p_1,p_2) :, : M_{--}(q_1,q_2) :] = N\delta(p_1+q_2)\delta(p_2+q_1), \tag{3.1}$$

thus the quadratic Hamiltonian gives a homogeneous equation of motion. Therefore we insist that *the interaction between two-body states has been hidden in the nontrivial algebra of the bilocal operators.* What we want to do is to construct an interacting meson theory for *finite* N. It would be more convenient if we can see the interaction manifest in the Hamiltonian. This will be achieved if we can express the two-body operators in terms of, say, some bosonic operators. Using bosonic operators is justified as follows.

For finite N theories, the real mesonic state may contain, in general, many-body components in addition to the two-body state;

$$|\mathrm{meson}>= |q\bar{q}> +|q\bar{q}q\bar{q}> +|q\bar{q}g> + \cdots, \tag{3.2}$$

where $|q\bar{q}>$ is a quark-antiquark two-body state, $|q\bar{q}q\bar{q}>$ is a four-body state, and so on. As was mentioned in the Introduction, in the context of the light-front Tamm-Dancoff approach [25, 26, 27], however, we know that in various models [18]-[24] the lowest meson consists of only a two-body component. Thus to regard the two-body state as a meson would not be so bad at least for the lowest mesonic excitation even for finite N theory. A general (color singlet) two-body state with its total momentum P may be given as the linear combination of $: M_{--}(p,q) : |0>$ with some normalization factor $\mathcal{N}$,

$$\begin{aligned} |q\bar{q}> &= \mathcal{N}\sum_i \int_0^P dk_-\phi(k_-) a^{i\dagger}_{k_-} d^{i\dagger}_{P-k_-}|0> \\ &= \mathcal{N}\int_0^P dk_-\phi(k_-) : M_{--}(-k_-, -P+k_-) : |0>, \end{aligned} \tag{3.3}$$

where $\phi(k)$ is the momentum space wave function of the relative motion between a quark and an antiquark (see the Appendix for the relation between M and the quark or antiquark operators). Even if this state corresponds to the lowest meson, it is not a bosonic state because of the nontrivial algebra (2.15). We cannot construct bosonic operators only from quark-antiquark operators. For massless $U(1)$ gauge theory, a regularized current operator satisfies bosonic commutators and it gives the usual bosonization; we obtain a Hamiltonian of a free massive boson. However, it contains $a^\dagger a$ and $d^\dagger d$ terms and is not a pure quark-antiquark operator.

The physical meaning of the non-bosonic properties of the two-body state is given as follows. Since the two-body state consists of fermions, the Pauli principle works and forbids the two-body states to be in the same state, for example the same momentum state. This effect works between two-body states. Indeed a single two-body state can be considered as a bosonic state, which is observed as

$$< 0|[: M_{++}(p_1, p_2) :, : M_{--}(q_1, q_2) :]|0 >= N\delta(p_1 + q_2)\delta(p_2 + q_1), \tag{3.4}$$

for arbitrary N. Hence the interaction between two-body states can be understood as the effect of the Pauli principle and thus this is a universal property of the fermion-pair systems. Note that the equation (3.1) can also be understood in the context of the Pauli principle; when $N \to \infty$, the number of allowed states for a fixed momentum goes to infinity and thus the Pauli principle loses its efficacy.

Since the two-body state is a good approximation for the lowest meson, we want to regard the state as a bosonic state. Indeed it behaves as a boson in the large N limit (3.1) and even for finite N, as long as there is a single two-body state (3.4). Therefore it is natural to try to find a representation where $: M :$ is expressed by some bosonic operator which is equivalent to the two-body state in these cases. If we are able to do this, we obtain a bosonic theory with interactions manifest in the Hamiltonian. As for the problem of how to represent operators whose algebra is nontrivial, in terms of bosonic operators, there is an answer as a standard technique in the many-body physics. It is the *boson expansion method*. Using this method, we can express the bilocal operators $: M :$ in terms of bosonic operators which correspond to $: M :$ in the large N limit. This method is introduced in the next section and applied to the algebra of $: M :$.

4 Application of boson expansion method

The boson expansion method [29] is one of the traditional techniques in non-relativistic many-body problems. In various systems we encounter a situation that there are some boson-like excitations even if the fundamental variables of the system are not bosons. The boson expansion method has been introduced for describing such bosonic excitations in non-bosonic systems. In this section we briefly review this method and apply it to our case, QCD_{1+1}.

4.1 Boson expansion method – a short review

Originally the boson expansion was invented by Holstein and Primakoff [31] to describe interacting spin waves in the Heisenberg ferromagnet. Although the first excitation from the

spin-aligned vacuum, i.e. a spin wave, is solved exactly, the next excitation is difficult to obtain due to the interaction between spin waves, which originates from the $SU(2)$ spin algebra. So they introduced an boson operator and represented the spin operators by it so that they satisfy the $SU(2)$ algebra. Then the theory is translated into a bosonic theory with interactions. A single boson state corresponds to the single spin wave. Later another representation of the spin algebra was proposed by Dyson [32]. Also, Usui [33] applied this method to plasma oscillations.

In nuclear physics, since bosonic excitations from the ground state are very important for understanding the collective vibrations, the boson expansion method has been investigated in great detail [29, 34, 35, 36]. It would be particle-hole operators that play an important role for describing (collective) bosonic excitations in the fermionic many-body problem. Let $a_\mu^\dagger$ be the creation operator of a *particle* with its quantum number μ and $d_i^\dagger$ that of a *hole* in state i. The four bifermion operators form an algebra $d_i a_\mu, a_\mu^\dagger d_i^\dagger, d_i^\dagger d_j$ and $a_\mu^\dagger a_\nu$;

$$\begin{aligned}
[d_i a_\mu, a_\nu^\dagger d_j^\dagger] &= \delta_{\mu\nu}\delta_{ij} - \delta_{\mu\nu} d_j^\dagger d_i - \delta_{ij} a_\nu^\dagger a_\mu, && (4.1)\\
[d_i a_\mu, a_\nu^\dagger a_\sigma] &= \delta_{\mu\nu} d_i a_\sigma, && (4.2)\\
[a_\mu^\dagger a_\nu, a_\sigma^\dagger a_\tau] &= \delta_{\nu\sigma} a_\mu^\dagger a_\tau - \delta_{\mu\tau} a_\sigma^\dagger a_\nu, && (4.3)\\
[d_i a_\mu, d_j^\dagger d_k] &= \delta_{ij} d_k a_\mu, && (4.4)\\
[d_i^\dagger d_j, d_k^\dagger d_l] &= \delta_{jk} d_i^\dagger d_l - \delta_{il} d_k^\dagger d_l. && (4.5)
\end{aligned}$$

Their first motivation is to obtain collective bosons for describing the collective excitation. The term "collective excitation" means that the number of the participating particles is large. Thus they usually start with collective fermion-pair operators such as $b_\mu^\dagger = \sum_{\nu i} C_{\nu i}^\mu a_\nu^\dagger d_i^\dagger$, and then represent these by bosonic operators [34]. Here however we do not follow the same path for reasons mentioned later and here we introduce the bosonic expressions for *pure* fermion-pair operators.

There are several ways to implement this algebra. Here we list three well-known representations. The different boson expansions below have been shown to be equivalent [37]. Let us begin with the Holstein-Primakoff (HP) type expansion. Introducing the boson operators,

$$[B_{\mu i}, B_{\nu j}^\dagger] = \delta_{\mu\nu}\delta_{ij}, \quad [B_{\mu i}, B_{\nu j}] = 0, \qquad (4.6)$$

the four operators are represented by

$$\begin{aligned}
a_\mu^\dagger a_\nu &= \sum_j B_{\mu j}^\dagger B_{\nu j} \equiv \mathcal{A}_{\nu\mu}, && (4.7)\\
d_k^\dagger d_l &= \sum_\mu B_{\mu k}^\dagger B_{\mu l}, && (4.8)
\end{aligned}$$

$$
\begin{aligned}
d_i a_\mu &= \sum_\nu (\sqrt{1-\mathcal{A}})_{\mu\nu} B_{\nu i}, && (4.9)\\
a_\mu^\dagger d_i^\dagger &= \sum_\nu B_{\nu i}^\dagger (\sqrt{1-\mathcal{A}})_{\mu\nu}^\dagger. && (4.10)
\end{aligned}
$$

These equations should be considered as follows; if we substitute the r.h.s. into the algebra (4.1)-(4.5), they hold by using the bosonic commutators (4.6). As long as we stay in the zero fermion-number sector, this replacement is correct. The square roots in (4.9) and (4.10) are defined by the corresponding Taylor series ($\mathcal{A}_{\mu\nu}^2 = \sum_\rho \mathcal{A}_{\mu\rho}\mathcal{A}_{\rho\nu}$). The term "expansion" comes from the fact that the HP type contains an infinite expansion in terms of the boson operator. There exists a subspace of full bosonic Fock space, where the square roots are well defined. This space is called the *physical subspace*. If we truncate the infinite series by a few terms, it would not be a good approximation as it is. This is why they consider the collective state first and apply the boson expansion to it instead of the pure bifermion operators.

Second, there is a finite expansion method, the Dyson expansion. In this method, the expressions (4.9) and (4.10) are replaced by $d_i a_\mu = B_{\mu i}$ and $a_\mu^\dagger d_i^\dagger = \sum_\nu B_{\nu i}^\dagger (1-\mathcal{A})_{\mu\nu}^\dagger$, respectively. While the expansion is finite, it is evident that the relation $(d_i a_\mu)^\dagger = a_\mu^\dagger d_i^\dagger$ does not hold. To remedy this, the Dyson expansion needs a particular method, which is called the unitary projection.

The third representation is the Schwinger type [38, 39], which is also finite. This method needs two kinds of bosons. In addition to $B_{\mu i}$, we introduce another boson A_{ij} which commutes with $B_{\mu k}$;

$$
\begin{aligned}
[A_{ij}, A_{kl}^\dagger] &= \delta_{ik}\delta_{jl}, \quad [A_{ij}, A_{kl}] = 0, && (4.11)\\
[A_{ij}, B_{\mu k}] &= [A_{ij}, B_{\mu k}^\dagger] = 0. && (4.12)
\end{aligned}
$$

Then the four operators are expressed as

$$
\begin{aligned}
a_\mu^\dagger a_\nu &= \sum_j B_{\mu j}^\dagger B_{\nu j}, && (4.13)\\
d_i^\dagger d_j &= \delta_{ij} - \sum_k A_{jk}^\dagger A_{ik}, && (4.14)\\
d_i a_\mu &= \sum_k A_{ik}^\dagger B_{\mu k}, && (4.15)\\
a_\mu^\dagger d_i^\dagger &= \sum_k B_{\mu k}^\dagger A_{ik}. && (4.16)
\end{aligned}
$$

It is evident from Eq. (4.14) that the state annihilated by these bosonic operators is not a true vacuum. The true vacuum state $|vac>_B$ is given as

$$
|vac>_B = \frac{1}{\sqrt{N!}} \sum_{P_h} (-1)^{P_h} P_h A_{1h_1}^\dagger A_{2h_2}^\dagger \cdots A_{Nh_N}^\dagger |0), \tag{4.17}
$$

where N is the number of the hole states, P_h means the permutation of the indices $\{h_1, h_2, \ldots h_N\}$, and $|0)$ is a state which satisfies $B_{\mu k}|0) = A_{ij}|0) = 0$.

4.2 Boson expansion method – application to QCD$_{1+1}$

Let us go back to two-dimensional QCD. The algebra we want to simplify is Eq. (2.15). Essentially this is the same as that of the fermion pair operators in nuclear physics, i.e. Eqs.(4.1)-(4.5). The particle-hole operators correspond to quark-antiquark operators, and so on. The only difference is the existence of an internal symmetry ($SU(N)$ or $U(N)$) and continuum indices (momentum). Taking these points into account, let us introduce the bosonic operators,

$$[B(p_1, p_2), B^\dagger(q_1, q_2)] = \delta(p_1 - q_1)\delta(p_2 - q_2), \tag{4.18}$$

$$[B(p_1, p_2), B(q_1, q_2)] = 0, \tag{4.19}$$

where momentum indices are defined to be positive ($p_i, q_i > 0$). Since $: M :$ is constructed so as to be color singlet, $B(p_1, p_2)$ need not carry color. In this paper we consider only HP type expansion. The bifermion operators are expressed as

$$: M_{-+}(p_1, p_2) : = \int_0^\infty dq \;\; B^\dagger(-p_1, q)B(p_2, q) \equiv \mathcal{A}(p_2, -p_1), \tag{4.20}$$

$$: M_{+-}(p_1, p_2) : = -\int_0^\infty dq \;\; B^\dagger(q, -p_2)B(q, p_1), \tag{4.21}$$

$$: M_{++}(p_1, p_2) : = \int_0^\infty dq \;\; (\sqrt{N - \mathcal{A}})(p_2, q)B(q, p_1), \tag{4.22}$$

$$: M_{--}(p_1, p_2) : = \int_0^\infty dq \;\; B^\dagger(q, -p_2)(\sqrt{N - \mathcal{A}})(q, -p_1). \tag{4.23}$$

Since $: M_{++} := O(N^{1/2})$ and $: M_{-+} := O(N^0)$, the bosonic operator is considered to be $O(N^0)$. Of course, the square roots in Eqs. (4.22) and (4.23) are defined by their Taylor expansions, for example,

$$\begin{aligned} : M_{++}(p_1, p_2) : &= \sqrt{N}B(p_2, p_1) \\ &- \frac{1}{2\sqrt{N}} \int_0^\infty dqdkB^\dagger(q, k)B(p_2, k)B(q, p_1) - \cdots . \end{aligned} \tag{4.24}$$

As was noted in the previous section, the bosonic representation should satisfy two conditions to keep in touch with the meson picture. The first requirement that $: M :$ itself becomes bosonic in the large N limit, i.e. Eq.(3.1)be satisfied as

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} : M_{--}(p_1, p_2) := B^\dagger(-p_1, -p_2). \tag{4.25}$$

And also the second one that the single two-body state should be bosonic for finite N is observed apparently as

$$: M_{--}(p_1, p_2) : |0 >= \sqrt{N}B^\dagger(-p_1, -p_2)|0), \tag{4.26}$$

where $|0)$ is a vacuum state of the bosonic operator; $B(p,q)|0) = 0$. That is why we used the HP type expansion.

Since the boson expansion becomes a good approximation for systems with large participating particles, they start with collective bosons instead of pure particle-hole operators in nuclear physics [34]. In our case, however, there is a parameter N associated with the internal symmetry and the bilocal operators are color singlet. The number of participating particles in the two-body state goes to infinity if $N \to \infty$. Thus we do not have to introduce collective operators at this stage. Collective bosons are introduced to give local bosons in the next section. Furthermore, note that also in the large N limit, the effect of the physical subspace vanishes. The existence of internal symmetry makes the situations simple.

Substituting (4.20)-(4.23) into the Hamiltonian (2.19), we obtain for the leading contribution;

$$\begin{aligned} H_{HP}^{(0)} &= -\frac{g^2 N}{8\pi} \int_0^\infty dp_1 dp_2 dq_1 dq_2 \; \delta(p_1 + p_2 - q_1 - q_2) \times \\ & \qquad \frac{\mathcal{P}}{(p_1 - q_2)^2} B^\dagger(p_1, p_2) B(q_2, q_1) \\ & \quad + \frac{1}{2}\left(m^2 - \frac{g^2 N}{2\pi}\right) \int_0^\infty dq dq' \; \left(\frac{\mathcal{P}}{q} + \frac{\mathcal{P}}{q'}\right) B^\dagger(q, q') B(q, q'), \end{aligned} \tag{4.27}$$

where the suffices (0) and HP imply the $O(N^0)$ contribution and the Holstein-Primakoff type expansion. The equation of motion is

$$\begin{aligned} i\partial_+ B(p, r-p) &= [B(p, r-p), H_{HP}^{(0)}] \\ &= -\frac{g^2 N}{4\pi} \int_0^r dq \frac{\mathcal{P}}{(p-q)^2} B(q, r-q) \\ & \quad + \frac{1}{2}\left(m^2 - \frac{g^2 N}{2\pi}\right)\left(\frac{\mathcal{P}}{p} + \frac{\mathcal{P}}{r-p}\right) B(p, r-p). \end{aligned} \tag{4.28}$$

Essentially this is equivalent to the 't Hooft equation (2.21).

The higher order Hamiltonians ($H_{HP}^{(n)} = O(N^{-n/2})$) are obtained by substituting the expansion of Eqs. (4.20)-(4.23). The next leading Hamiltonian is on the order of $N^{-1/2}$ and is given, after normal ordering, by

$$\begin{aligned} H_{HP}^{(1)} &= \frac{g^2\sqrt{N}}{4\pi} \int_0^\infty dp_1 dp_2 dq_1 dq_2 dk \;\; \delta(p_1 - p_2 + q_1 + q_2) \;\; \frac{\mathcal{P}}{(p_1 + q_2)^2} \times \\ & \quad \left\{ B^\dagger(k, p_2) B(k, p_1) B(q_2, q_1) - B^\dagger(p_2, k) B(p_1, k) B(q_1, q_2) + \text{h.c.} \right\}, \end{aligned} \tag{4.29}$$

which corresponds to a three-point vertex.

The expressions (4.20)-(4.23) are very similar to those of Nakamura and Odaka [4]. They introduced bosonic operators as the leading contribution of $:M:$ in the $1/N$ expansion. This

construction is the same as the requirement (4.25) and indeed their bosonic operator is equivalent to ours in the large N limit. However, if we express their representation in our notation, it is different from ours;

$$\begin{aligned}
:M_{-+}(p_1,p_2): &= \int_0^\infty dq \;\; B^\dagger(-p_1,q)B(p_2,q), \\
:M_{+-}(p_1,p_2): &= -\int_0^\infty dq \;\; B^\dagger(q,-p_2)B(q,p_1) \equiv \mathcal{C}(p_2,-p_1), \\
:M_{++}(p_1,p_2): &= \int_0^\infty dq \;\; (\sqrt{N+\mathcal{C}})(p_2,q)B(q,p_1), \\
:M_{--}(p_1,p_2): &= \int_0^\infty dq \;\; B^\dagger(q,-p_2)(\sqrt{N+\mathcal{C}})(q,-p_1).
\end{aligned}$$

This is one of the HP type expansions. They were not aware that this is the boson expansion methods. However, once we recognize the usefulness of the boson expansion method, we can apply this idea to many other models even with small N. Other *finite* expansion methods such as the Dyson type or methods using collective fermion pairs will be found to be more useful for such theories.

5 Local boson operators as collective states

It was bilocal operators that we introduced in the previous section. Here we construct local boson operators. The following is essentially the same as that of Nakamura and Odaka [4]. However, according to the many-body physics, their construction is understood as the introduction of the collective states from $B(p,q)$. Collective boson operators are defined as the linear combination of bilocal bosons with their total momentum r,

$$b_r^\dagger = \sqrt{r}\int_0^1 dz \;\; \phi(z)B^\dagger(rz,(1-z)r). \tag{5.1}$$

Let us assume that the wave function $\phi(z)$ forms an orthonormal complete set;

$$\int_0^1 dz \;\; \phi_n^*(z)\phi_m(z) = \delta_{nm}, \tag{5.2}$$

$$\sum_n \phi_n(z)\phi_n^*(z') = \delta(z-z'). \tag{5.3}$$

As far as $\{\phi_n\}$ is a complete set, the operators

$$b_r^{(n)\dagger} = \sqrt{r}\int_0^1 dz \;\; \phi_n(z)B^\dagger(rz,(1-z)r), \tag{5.4}$$

$$b_r^{(n)} = \sqrt{r}\int_0^1 dz \;\; \phi_n^*(z)B(rz,(1-z)r), \tag{5.5}$$

satisfy the bosonic commutators

$$[b_r^{(n)}, b_{r'}^{(m)\dagger}] = \delta_{nm}\delta(r-r'), \quad [b_r^{(n)}, b_{r'}^{(m)}] = 0. \tag{5.6}$$

Let us determine $\{\phi_n\}$ so that b and $b^\dagger$ diagonalize the lowest Hamiltonian $H_{HP}^{(0)}$. Inserting these operators, the equation of motion can be written as

$$\mu_n^2 \phi_n(z) = \left(m^2 - \frac{g^2 N}{2\pi}\right)\left(\frac{1}{z} + \frac{1}{1-z}\right)\phi_n(z) - \frac{g^2 N}{2\pi}\int_0^1 dz' \frac{\phi_n(z')}{(z-z')^2}, \tag{5.7}$$

which means that the wave function $\phi_n(x)$ is a solution of the 't Hooft equation. If we define a scalar field by

$$\Phi_n(x) \equiv \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dr_-}{\sqrt{2r_-}} \left(b_r^{(n)} e^{-irx} + b_r^{(n)\dagger} e^{irx}\right), \tag{5.8}$$

this is a field operator of the n-th excitation state. $\Phi_n(x)$ satisfies the usual LF commutator;

$$[\Phi_n(x),\ \Phi_m^\dagger(y)]_{x^+=y^+} = -\frac{i}{4}\delta_{nm}\epsilon(x^- - y^-). \tag{5.9}$$

Using this operator, $H_{HP}^{(0)}$ is diagonalized and can be written as

$$\begin{aligned} H_{HP}^{(0)} &= \sum_n \int_0^\infty dr\ \frac{\mu_n^2}{2r}\ b_r^{(n)\dagger} b_r^{(n)} \\ &= \sum_n \int_{-\infty}^\infty dx\ \frac{\mu_n^2}{2} : \Phi_n(x)^2 : . \end{aligned} \tag{5.10}$$

This is the Hamiltonian of free scalar bosons with their mass being the eigenvalues of the 't Hooft equation. The next order Hamiltonian is

$$H_{HP}^{(1)} = \frac{g^2\sqrt{N}}{4\pi}\int_0^\infty \frac{dR}{\sqrt{R}}\int_0^1 dz \sum_{nml}\left(K_{nml}(z) - \tilde{K}_{nml}(z)\right)\left(b_R^{(n)\dagger} b_{Rz}^{(m)} b_{R(1-z)}^{(l)} + \text{h.c.}\right), \tag{5.11}$$

where

$$\begin{aligned} K_{nml}(z) &= \int_0^1 dx \int_0^1 dy \frac{\sqrt{z(1-z)}}{\{z(1-x)+(1-z)y\}^2}\ \ \phi_n^*(zx)\phi_m(x)\phi_l(y), \\ \tilde{K}_{nml}(z) &= \int_0^1 dx \int_0^1 dy \frac{\sqrt{z(1-z)}}{\{zx+(1-z)(1-z)\}^2}\ \ \phi_n^*(1-z(1-x))\phi_m(x)\phi_l(y). \end{aligned}$$

Hence for finite N, we obtain a Hamiltonian of infinite kinds of bosons interacting with each other. In principle, there are infinitely many-point vertices which originate from the infinite expansion of the HP type.

Recently Barbón and Demeterfi [9] derived an effective Hamiltonian in the approximation that the following operators are bosonic;

$$\begin{aligned} \alpha_P^\dagger &\equiv \mathcal{N}\int_0^P dk_- \phi(k_-) : M_{--}(-k_-, -P+k_-) : \ \ , \end{aligned} \tag{5.12}$$

$$\begin{aligned} \alpha_P &\equiv \mathcal{N}\int_0^P dk_- \phi^*(k_-) : M_{++}(P-k_-, k_-) : \ \ . \end{aligned} \tag{5.13}$$

Indeed they satisfy $[\alpha_P, \alpha_Q^\dagger] = \delta(P-Q) + O(1/N)$ and thus are bosonic in the large N limit, which has been insisted on many times. It is characteristic that their Hamiltonian has only finite terms in contrast to ours. This is clearly because they did not incorporate the next leading order of the commutator. Their method cannot be extended to small N theories, while the boson expansion methods have other ways to avoid it, as has been noted.

6 Conclusion and Discussions

In this paper we have found that the boson expansion methods are useful for constructing bosonic theories from two dimensional QCD on the light-front. In this method, fermion bilinear operators can be nonlinearly expressed by bilocal bosonic operators such that the same algebra holds. Among various representations, if we choose the Holstein-Primakoff type expansion, the bosons are identified with mesons in the large N limit, and for finite N we obtain an interacting boson theory. The physical meaning of this procedure is as follows. Because of the inevitable effects of the fermion statistics, the two-body states cannot stay bosonic. This effect is called the kinematical effect because it is independent of the Hamiltonian. The kinematical complexity in the two-body states is translated into interactions among bosons via the boson expansion methods. Furthermore, we can build local boson operators from bilocal bosons as the collective state. Since this is determined to make the lowest Hamiltonian diagonal, this effect is called the dynamical effect. Eventually we have obtained a local interacting theory with infinite kinds of bosons. Thanks to the existence of the parameter N, we can argue the above two effects separately, which is not the case in nuclear physics.

In fact, the introduction of the gauge invariant operator was not crucial for implementing the boson expansion method, but it has made the argument very transparent and leads to the discovery of the applicability of the boson expansion methods. Although there are no dynamical degrees of freedom in the gauge field when the space is infinite, if we work in finite (light-front) space, there exist dynamical zero modes of the gauge field. The gauge-invariant operators contain them which is not the case in nuclear physics. The boson expansion method can also be applied to these operators as far as the winding of the zero mode is trivial.

The application of the boson expansion method has been performed for light-front field theory. However, we can do the same things for perturbation theory in an equal-time formulation. The merit on the light-front is, thanks to the triviality of the vacuum, that the few-body states are expected to be good variables for describing hadrons. This is true even for small N theories. Thus we expect that bosons also describe hadrons. In equal-time formulations, however, we cannot expect in general that only the two-body state gives a meson. Thus boson expansion

methods are expected to work better on the light-front than on the equal-time.

There are many things to be clarified in this formalism. Especially, it is important to know the relation to the usual bosonization. In bosonization, the number of bosons is finite, while it is infinite in our formalism. As a more concrete problem, the existence of meson-meson bound states is suggested in Ref. [23]. It would be interesting to investigate it in our framework. Furthermore, there is the problem of how we can describe baryons. Recently Rajeev [5] argued that the bilocal operator $M(p,q)$ is an element of an infinite dimensional Grassmannian, and that the baryon number is understood as the topological invariant (virtual rank) of the Grassmannian, and thus baryons are interpreted as solitons. One of our motivations was to see how we can obtain an effective theory such as the Skyrme model. In the Skyrme model, a baryonic state is considered as a soliton in the large N limit [40]. Our resulting Hamiltonian is a *nonlinear* theory of mesons. Although the topological effect is difficult to observe in the trivial vacuum of the light-front, it is intriguing to find a solitonic configuration in our formalism.

In nuclear physics, the boson expansion method is considered as a method that goes beyond the Tamm-Dancoff approximation. Since the structure of the LF theories is similar to non-relativistic quantum theory, it seems to be natural that we have found that the boson expansion method is also applicable to LF field theory when we want to go beyond the TD approximation. Also, in many-body physics, the boson introduced is thought to be appropriate for describing the Nambu-Goldstone boson when there is symmetry breaking. Indeed at the very first application, the spin wave corresponds to the NG boson under spontaneous magnetization. Therefore there might be a possibility that also on the LF the boson introduced in the boson expansion methods can describe the NG boson. Since this is a very important problem, we should apply the boson expansion method to higher dimensional theories where the NG boson or the pseudo NG boson will exist.

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Appendix

If we use the fermion operator after solving the constraints, we can easily relate the bilocal operators $:M:$ to the quark or anti-quark operators. As in eq.(2.11), the usual definition of

the quark or anti-quark creation/annihilation operators is given by

$$\psi_i(x) = \int_0^\infty \frac{dk_-}{\sqrt{2\pi}} \left\{ a^i_{k_-} e^{-ikx} + d^{i\dagger}_{k_-} e^{ikx} \right\}, \tag{A.1}$$

where the momentum takes only positive values $k_- > 0$. On the other hand, Fourier transformation of the gauge invariant operators is given as in eq. (2.13). Thus its momenta can take negative values. Taking care of this point, the relation between two is given as follows:

$$:M_{++}(p,q): \;=\; d^i_p a^i_q \tag{A.2}$$

$$:M_{--}(p,q): \;=\; a^{i\dagger}_{-p} d^{i\dagger}_{-q} \tag{A.3}$$

$$:M_{+-}(p,q): \;=\; -d^{i\dagger}_{-q} d^i_p \tag{A.4}$$

$$:M_{-+}(p,q): \;=\; a^{i\dagger}_{-p} a^i_q \tag{A.5}$$

From these we can see that $:M_{++}:$ is the hermitian conjugate to $:M_{--}:$,

$$(:M_{++}(p,q):)^\dagger =: M_{--}(-q,-p): \quad . \tag{A.6}$$

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